Junior Scientist Workshop: Theoretical Neuroscience

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1 Robustness in neural computation

Neural networks undergo structural changes throughout their existence, whether through developmental processes, learning experiences, or even damage. The concept of robustness is vital in neural computation as it allows the brain to maintain stable and reliable performance despite perturbations and noise. This tutorial proposal aims to introduce theoretical neuroscientists to tools to assess the robustness properties of models of neural computation. Participants will gain a comprehensive understanding of how persistence analysis can shed light on the stability and reliability of neural networks.

Structural stability analysis can be employed to investigate how neural networks respond to damage or lesions. By assessing the network's ability to retain functionality and stability in the presence of structural disruptions, researchers can understand the mechanisms of brain resilience and recovery. Additionally, Normally Hyperbolic Invariant Manifolds (NHIMs) provide a framework for comprehending the dynamics and stability of complex systems like neural networks by identifying the key structures that govern their behavior. NHIMs act as critical structures that maintain the system's stability, enabling robust neural computation.

Finally, Fenichel's persistence theorem provides a valuable tool for analyzing the long-term behavior and robustness of dynamical systems, including those encountered in theoretical neuroscience. Fenichel's persistence theorem states that if a dynamical system has a stable manifold associated with a particular equilibrium point or steady state, then the solutions that start sufficiently close to this manifold persistently stay in its vicinity despite small perturbations. The tutorial will discuss practical consequences of this theorem for models of neural computation.

1.1 Introduction

Robustness in dynamical systems refers to the ability of a system to maintain its functionality or desired behavior even in the presence of external disturbances or parameter variations. Bifurcations are key indicators of the system's robustness.

When a system undergoes a bifurcation, it may transition from having a single (asymptotically) stable fixed point stable to an unstable state, or vice versa. The ability of a system to retain stability is a measure of its robustness.

Identifying bifurcation points and understanding how they affect system behavior is critical for assessing the robustness of dynamical systems, particularly in engineering, biology, and other fields where stability and predictability are paramount.

2 Parameter variability in biological systems

Biological Variability: In living organisms, individual neural networks can exhibit considerable variability in terms of neuron properties, synaptic strengths, and connectivity patterns. This variability arises due to genetic differences, environmental factors, and developmental processes. As a result, the specific values of key parameters (e.g., synaptic weights, time constants, thresholds) can vary from one individual to another and even within the same individual over time, while the behavior is the same.

Noise: Neural networks are subject to intrinsic and extrinsic sources of noise. These sources of variability can affect the precise values of parameters and influence network behavior.

Plasticity and Adaptation: Biological neural networks are highly adaptive. They can change their connectivity and functional properties in response to learning, experiences, and environmental changes. The dynamic nature of these networks means that parameters like synaptic weights and neural excitability can be continuously adjusted and are therefore uncertain in the short term.

3 Bifurcation

In the presence of parameter uncertainty, it's essential to design systems that remain *functional* and *stable* despite variations in parameter values. Understanding bifurcations enables us to assess the system's response to parameter uncertainties. A bifurcation is a critical parameter value where the qualitative behavior of the system undergoes a significant change as a parameter is varied. In other words, it's a point at which a system transitions from one type of behavior to another. Bifurcations can involve the creation or destruction of equilibrium points, limit cycles, or chaotic behavior. They are fundamental to understanding how dynamical systems respond to changes in parameters. Bifurcations are intimately connected to structural stability because they often represent the boundary between structural stability and instability. By identifying regions in parameter space where bifurcations occur, we can take steps to ensure that the system remains robust by avoiding undesirable behaviors associated with bifurcation points.

3.1 Saddle-node bifurcation

A saddle-node bifurcation is a collision and disappearance of two equilibria in dynamical systems. See also Sec. 6.1.

Normal Form:

$$\dot{x} = \mu + x^2 \tag{1}$$

The equilibria points only exist at the solution to the quadratic equation $x(t) = \pm\sqrt{-\mu}$. Now notice that we have three scenarios,

1. If $\mu < 0$ we have that there exists two equilibrium points, one at $-\sqrt{-\mu}$ and one at $\sqrt{-\mu}$. Notice furthermore, the stable equilibrium point is at the negative value and unstable at the positive.

2. If $\mu = 0$ we see that we only have one equilibrium (at the origin), however this equilibrium is upon a saddle point, therefore unsurprisingly we call it a *saddle node bifurcation*,

3. If $\mu > 0$ we see that x can only be imaginary meaning that there are no equilibria on the real domain, therefore we have no equilibria.

3.2 SNIC

Saddle-node bifurcation on invariant circle, also known as SNIC or SNLC (saddle-node on limit cycle) bifurcation, occurs when the center manifold of a saddle-node bifurcation forms an invariant circle. Such a saddle-node homoclinic bifurcation results in the birth of a limit cycle when the saddle-node disappears. The period of this cycle tends to infinity as the parameter approaches its bifurcation value. It plays an important role in computational neuroscience, where it is exhibited by Class 1 excitable systems (such as cortical pyramidal neurons).

The theta model is the normal form for the saddle-node on a limit cycle bifurcation (SNIC).

$$\frac{d\theta}{dt} = 1 - \cos\theta + (1 + \cos\theta)I(t) \tag{2}$$

where $I(t)$ is the input to the model. The variable θ lies on the unit circle and ranges between 0 and 2π . When $\theta = \pi$ the neuron "spikes", that is, it produces an action potential.

The bifurcation occurs as a parameter varies through the critical value of I=0.

1. When $I < 0$ there is a pair of equilibria. One of the equilibria is a saddle point with a one-dimensional unstable manifold.

2. When $I = 0$ there is a saddle node with a homoclinic orbit.

3. For $I > 0$ there are no equilibria.

4 Structural stability

Structural stability is a fundamental concept in the study of dynamical systems. A structurally stable system is one for which small perturbations in the system's parameters do not lead to drastic qualitative changes in the system's behavior. In other words, if a system is structurally stable, its behavior remains robust under perturbations.

Robustness is closely related to the notion of structural stability. In structurally stable systems, perturbations result in smooth changes in the system's behavior. However, when a system loses structural stability at a bifurcation point, small parameter changes can lead to drastical qualitative changes in the system's dynamics. Understanding the types and mechanisms of bifurcations is essential for assessing the structural stability of a dynamical system.

Definition 1 (Flow of an autonomous system of ODEs)**.** Let $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ be a (time-independent) vector field and $\mathbf{x}\colon \mathbb{R} \to \mathbb{R}^n$ the solution of the initial value problem

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t)), \quad \mathbf{x}(0) = \mathbf{x}_0.$$

Then $\phi(\mathbf{x}_0, t) = \mathbf{x}(t)$ is the flow of the vector field $\mathbf{F}$. We will use the notation $\mathcal{O}(x, \phi) \coloneqq \{\phi(x, t) \mid t \in \mathbb{R}\}$ to denote an orbit.

Definition 2 (Closed orbit)**.** A *closed orbit* of a flow is a solution $\mathbf{x}(t)$ such that there exists $t_1, t_2 \in \mathbb{R}$ with $t_1 \neq t_2$ for which $\mathbf{x}(t_1) = \mathbf{x}(t_2)$.

Definition 3 (Topological equivalence)**.** Let X, Y be two topological spaces. Let ϕ be a flow on X, and ψ be a flow on Y. Then ϕ and ψ are *topologically equivalent* if there is a homeomorphism $h\colon X \to Y$ mapping orbits of ϕ to orbits of ψ homeomorphically, and preserving the orientation of the orbits. In other words, there must be an increasing map $\tau\colon X \times \mathbb{R} \to \mathbb{R}$ such that

$$h(\phi_{\tau(t,x)}(x)) = \psi_t(h(x)). \tag{3}$$

Theorem 1 (One Dimensional Equivalence)**.** *Two flows and in are topologically equivalent iff their equilibria, ordered on the line, can be put into one-to-one correspondence and have the same topological type (sink, source or semistable).*

4.1 Dynamical systems on manifolds

For a more general setting one might study flows on manifolds.

Definition 4. Let M be a compact Riemannian manifold without boundary. For each $r \geq 1$, let $\mathcal{X}^r(M)$ denote the set of C^r vector fields of M, endowed with the C^r topology. Every $S \in \mathcal{X}^r(M)$ generates a C^r flow $\phi = \phi^S\colon M \times \mathbb{R} \to M$. Two flows are topologically equivalent if there is a homeomorphism $h\colon M \to M$ that maps the orbits of one flow onto those of the other flow while preserving the orientation. We say S is C^r structurally stable if S has a C^r neighborhood $\mathcal{U}$ in $\mathcal{X}^r(M)$ such that every $X \in \mathcal{U}$ generates a flow ϕ^X that is topologically equivalent to ϕ^S (i.e. it sends the orbits of X to the orbits of Y, preserving the orientation of the orbits.

Remark 1. For every manifold, structurally stable flows form non-empty open subsets of $\mathcal{X}^r(M)$ [4].

Remark 2. If a set is equipped with a topology and an equivalence relation then its structurally stable elements are those interior to the equivalence classes. The "structure" is whatever is preserved by the equivalence relation; its structure remains the same when a structurally stable element is perturbed. For discrete dynamical systems the set is $D = \mathrm{Diffeo}(M)$, equipped with the C^1 topology, and the equivalence relation is topological conjugacy. For flows the space is X and the equivalence relation is orbit equivalence. Discrete dynamical systems and flows are actions by the groups $\mathbb{Z}$ and $\mathbb{R}$, respectively. For actions of more general groups the equivalence relation is similar: orbits are sent to orbits by a homeomorphism.

4.2 Structural stability in the plane

With the definition of structural stability, we can state the theorem that describes which systems are structurally stable. The theorem describes requirements for the system's recurrent sets. An orbit is recurrent if it is contained in its own ω-limit set or its own α-limit set.

Theorem 2 ([5])**.** *A necessary and sufficient condition for the system*

$$\frac{dx}{dt} = f(x), \quad x \in \mathbb{R}^2 \tag{4}$$

to be structurally stable *is that its flow has* hyperbolic periodic orbits*,* no other recurrence*, and* no saddle connections *(there is no trajectory connecting saddle points).*

4.2.1 Hyperbolicity

The fixed point x of the flow $\phi_t \colon M \to M$ is hyperbolic if x is a hyperbolic fixed point of the diffeomorphism $\phi_1 : M \to M$. An alternate way of saying this is as follows: If x is a fixed point of the flow $\phi_t \colon M \to M$, then the derivative $D\phi_t(x) \colon T_x(M) \to T_x(M)$ defines a linear representation of the real line and so can be written in the form $D\phi_t(x) = e^{tA}$ where A is a linear endomorphism of $T_x(M)$.

Hyperbolicity for $M = \mathbb{R}^n$ is an eigenvalue condition. The eigenvalues of the linear part of a vector field at a hyperbolic singularity have nonzero real part. In the case of a closed orbit $\gamma = \mathcal{O}(x, \phi)$, we choose a point $p \in \gamma$ and a transversal τ to γ at p. The vector field's flow defines a local diffeomorphism (the first-return map) of τ to itself having p as a fixed point. Hyperbolicity of γ means that the eigenvalues of the linear part of the first return map at p have nonzero real part.

Hyperbolicity of an invariant manifold can be defined in terms of stable and unstable subspaces. Let M be an invariant manifold, and let $\mathrm{T}M$ denote the tangent space at M. Hyperbolicity implies that there exist complementary subspaces E^s and E^u of T_M such that for all $x \in M$, the linearization of the flow at x has the following properties:

- Stable Subspace: For all $v \in E^s$, the linearized flow contracts in the E^s direction, i.e.,
$$\lim_{t\to\infty} \frac{\|D\Phi_t(x)v\|}{\|v\|} = 0.$$
- Unstable Subspace: For all $u \in E^u$, the linearized flow expands in the E^u direction, i.e.,
$$\lim_{t\to-\infty} \frac{\|D\Phi_t(x)w\|}{\|w\|} = 0.$$

Example 3. An equilibrium is a sink if all the eigenvalues of the linearization are negative, a source if all of them are positive, and a saddle otherwise.

Fun Fact: The Hartman–Grobman theorem states that the orbit structure of a dynamical system in a neighbourhood of a hyperbolic equilibrium point is topologically equivalent to the orbit structure of the linearized dynamical system.

4.3 Structural stability in dimensions greater than two

Theorem 4 ([7], [8])**.** *A vector field S is structurally stable if it satisfies Axiom A and the Strong Transversality conditions.*

4.3.1 Axiom A

A point $x \in M$ is *nonwandering* of the vector field S if for any neighborhood V of x in M, there is $t \geq 1$ such that $\phi_t(V) \cap V \neq \emptyset$. The set of *nonwandering* points of S is the nonwandering set of S, and denoted by $\Omega(S)$.

Definition 5. S is an axiom A flow if the following two conditions hold:

1. The nonwandering set of $S, \Omega(S)$, is a hyperbolic set and compact.
2. The set of periodic points of S is dense in $\Omega(S)$.

Singularities (i.e. points where $S(x) = 0$) and points of periodic orbits are all nonwandering.

Definition 6. Suppose M is a manifold, $\phi_t \colon M \to M$ is a flow. We say that ϕ_t is *hyperbolic* if for every $p \in M$ there is a splitting of the tangent space $\mathrm{T}_pM = E^s(x) \oplus E^0(x) \oplus E^u(x)$, where $E^0 = \langle \dot{\phi}_t \rangle$ is the flow direction and there are constants $C > 0$ and $\lambda \in (0, 1)$ such that for every $t > 0$ one has
$$\|D\phi_t(v)\| \leq C\lambda^t \|v\|$$
for $v \in E^s(x)$ and
$$\|D\phi_{-t}(v)\| \leq C\lambda^t \|v\|$$
for $v \in E^u(x)$.

4.3.2 Strong Transversality Condition

If a vector field S satisfies Axiom A, then for any $x \in M$, the stable manifold

$$W^s(x) = \left\{ y \in M \colon \lim_{t \to +\infty} d(\phi_t(y), \phi_t(x)) = 0 \right\}$$

of x and the unstable manifold

$$W^u(x) = \left\{ y \in M \colon \lim_{t \to -\infty} d(\phi_t(y), \phi_t(x)) = 0 \right\}$$

of x are each an injectively immersed C^r submanifold of M, if S is C^r.

An Axiom A system S satisfies the strong transversality condition if $W^s(x)$ is transverse to $W^u(x)$ at all $x \in M$. Roughly, this requires that the stable and unstable manifolds cross when they intersect.

Two submanifolds M_1 and M_2 are transverse if their tangent spaces span $\mathbb{R}^n$.

Remark 3. The stability and genericity of transversality make it a very powerful condition, and give rise to a number of applications, in many branches of science which might not initially seem related to differentiable manifolds. If a data set can be represented as a manifold, which is transverse to some condition that we care about, then we know that any (small) perturbations of the data set will not effect its relation to this important condition.

Remark 4. For higher than 2 dimensional systems saddle-to-saddle connections can be structurally stable, and can be hence used for robust neural computation with transients, see also [6].

5 Persistence of Normally Hyperbolic Invariant Manifolds

The following sections are based on Chapter 1 and 2 in [3] and Chapter 1 in [2].

NHIMs are robust structures that persist under small perturbations of the system's parameters. This robustness makes them valuable tools for analyzing the long-term behavior of dynamical systems, especially in applications where parameter variations are common. They provide a structured framework for analyzing the local stability and global behavior of solutions in the vicinity of certain invariant sets, such as equilibrium points or periodic orbits, in a dynamical system defined by differential equations. This section delves into the theory of NHIMs for flows, highlighting their significance and mathematical properties. NHIMs play a pivotal role in understanding the qualitative dynamics of a system. They provide a structured framework for analyzing the local stability and global behavior of solutions in the vicinity of certain invariant sets, such as equilibrium points or periodic orbits, in a dynamical system defined by differential equations.

Remark 5. NHIM results and fast-slow decomposition has been applied in neuroscience, for example to reduce the dimensionality of the original Hodgkin-Huxley model [**?**] to two dimensions [**?**] and to further simplify it as the theta neuron model [**?**].

A Normally Hyperbolic Invariant Manifold (NHIM) is a subset of the state space of a dynamical system that possesses two primary properties: **invariance** and **normal hyperbolicity**.

5.1 Invariance

The manifold remains invariant under the dynamics of the system, meaning that if a solution trajectory starts on the manifold, it stays on the manifold for all future times.

Definition 7. Let M be a compact connected C^r-manifold with boundary embedded in $\mathbb{R}^N$. Let $\phi_t(\cdot)$ denote the flow defined by the vector field

$$\dot{z} = H(z) \text{ for } z \in \mathbb{R}^N, H \in C^r(\mathbb{R}^N, \mathbb{R}^N) \text{ with } r \geq 1 \tag{5}$$

- invariant manifold: if for every $p \in M$, we have $\phi_t(p) \in M$ for all $t \in \mathbb{R}$.
- inflowing invariant manifold: if for every $p \in \partial M$, the vector field is pointing strictly inward and for all $p \in M, \phi_t(p) \in M$ for all $t \geq 0$.
- overflowing invariant manifold: if for every $p \in \partial M$, the vector field is pointing strictly outward and for all $p \in M, \phi_t(p) \in M$ for all $t \leq 0$.
- locally invariant manifold: if for each $p \in M$, there exists a time interval $I_p = (t_1, t_2)$ such that $0 \in I_p$ and $\phi_t(p) \in M$ for all $t \in I_p$. Local invariance means that trajectories can enter or leave M only through its boundaries.

Remark 6. Note that reversing the time direction in Eq.5 turns an overflowing invariant manifold into an inflowing invariant manifold and conversely. Therefore, we shall restrict to overflowing invariant manifolds for the remaining discussion.

5.2 Normal hyperbolicity

The behavior of solutions near the manifold is characterized by a combination of stable and unstable directions. More precisely, the tangent space of the manifold can be decomposed into three subspaces: the stable subspace, the unstable subspace, and the center subspace. The stable and unstable subspaces are responsible for the attracting and repelling behavior of solutions near the manifold, respectively, while the center subspace accounts for any other directions.

5.3 Fast–Slow Systems

Definition 8. A *fast–slow vector field* (or (m,n)-fast–slow system) is a system of ordinary differential equations taking the form

$$\begin{aligned} \epsilon \tfrac{dx}{d\tau} &= \epsilon\dot{x} = f(x,y,\epsilon), \\ \tfrac{dy}{d\tau} &= \dot{y} \;\; = g(x,y,\epsilon), \end{aligned} \tag{6}$$

where $f\colon \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$, $g\colon \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, and $0 < \epsilon \ll 1$. Furthermore, the x variables are called fast variables, and the y variables are called slow variables.

Setting $t = \tau/\epsilon$ gives the equivalent form

$$\begin{aligned} \tfrac{dx}{dt} &= x' = f(x,y,\epsilon), \\ \tfrac{dy}{dt} &= y' = \epsilon g(x,y,\epsilon). \end{aligned} \tag{7}$$

The parameter ϵ can be thought of as the "separation" of time scales and is sometimes called the time-scale parameter. If it appears in a statement or theorem, then it indicates that ϵ is sufficiently small, i.e., $0 < \epsilon \ll 1$ means that there exists some $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0]$, the statement of the theorem holds.

The differential-algebraic equation obtained by setting $\epsilon = 0$ in the formulation of the slow time scale is called the slow subsystem or slow vector field:

$$\begin{aligned} 0 &= f(x,y,0), \\ \dot{y} &= g(x,y,0). \end{aligned} \tag{8}$$

The flow generated by Eq. 8 is called the slow flow.

The slow subsystem is also referred to as the reduced problem and its flow as the reduced flow. Note that Eq. 8 is not an ODE, but an ODE with an algebraic constraint $f(x,y,0) = 0$. Therefore, we have a differential-algebraic equation (DAE). Initial conditions $x(0) = x_0$ and $y(0) = y_0$ must satisfy the constraint for solutions to exist.

The parameterized system of ODEs obtained by setting $\epsilon = 0$ on the fast time scale formulation Eq. 7 is called a fast subsystem or fast vector field:

$$\begin{aligned} x' &= f(x,y,0), \\ y' &= 0. \end{aligned} \tag{9}$$

The flow of Eq. 9 is called the fast flow.

The set of equations Eq. 7 is also referred to as the layer equations or the layer problem. This terminology encodes the geometric idea that each fixed y describes one "layer" of the fast subsystem.

The case $\epsilon = 0$ is also called the singular limit. The slow and fast formulations give a hint as to how we should analyze the full system with $0 < \epsilon \ll 1$.

Definition 9. The *critical set* is defined as:

$$C_0 = \{(x,y) \in \mathbb{R}^m \times \mathbb{R}^n \colon f(x,y,0) = 0\}.$$

If C_0 is a submanifold of $\mathbb{R}^m \times \mathbb{R}^n$, we refer to C_0 as the critical manifold.

Proposition 1. *Equilibrium points of the fast flow are in one-to-one correspondence with points in* C_0*.*

Definition 10. A subset $S \subset C_0$ is called normally hyperbolic if the $m \times m$ matrix $(D_x f)(p, 0)$ of first partial derivatives with respect to the fast variables has no eigenvalues with zero real part for all $p \in S$.

Proposition 2. *A subset* $S \subset C_0$ *is normally hyperbolic if and only if for each* $p = (x^*, y^*) \in S$*, we have that* x^* *is a hyperbolic equilibrium point of* $x' = f(x, y^*, 0)$*.*

Let's connect the general theory of normally hyperbolic invariant manifolds from Sec. 5.5 to fast–slow systems following [1]. The first question is, when is a critical manifold C_0 normally hyperbolic in the context of fast–slow systems?

Theorem 5 (Fenichel's theorem, fast-slow version [1])**.** *Suppose* $S = S_0$ *is a compact normally hyperbolic submanifold (possibly with boundary) of the critical manifold* C_0 *of Eq. 6 and that* $f, g \in C^r (r < \infty)$*. Then for* $\epsilon > 0$ *sufficiently small, the following hold:*

- *F1 There exists a locally invariant manifold* S_ϵ *diffeomorphic to* S_0*.*
- *F2* S_ϵ *is normally hyperbolic and has the same stability properties with respect to the fast variables as* S_0 *(attracting, repelling, or of saddle type).*
- *F3* S_ϵ *has Hausdorff distance* $O(\epsilon)$ *to* S_0 *(as* $\epsilon \to 0$*).*
- *F4 The flow on* S_ϵ *converges to the slow flow as* $\epsilon \to 0$*.*
- *F5* S_ϵ *is* C^r*-smooth.*

5.3.1 The Slow Flow

The next goal is to obtain an analytical expression for the slow flow on the critical manifold C_0.

Suppose now that C_0 is a manifold and $p \in C_0$ is a regular point. Then the implicit function theorem yields the existence of a map $h\colon \mathbb{R}^n \to \mathbb{R}^m$ describing C_0, locally near p, as a graph, and $f(h(y), y, 0) = 0$ holds near p. The map h can be used to reduce the slow subsystem $0 = f(x, y, 0), \dot{y} = g(x, y, 0)$, to the simpler form $\dot{y} = g(h(y), y, 0)$.

5.4 Generalized Lyapunov-type numbers

For a definition of normal hyperbolicity (similar to the concept of hyperbolicity) we need constraints on the speed of the flow and the ratio of the speeds in the normal and tangent direction along the invariant manifold. This is measured with the generalized Lyapunov-type numbers. Let $\Pi\colon \mathrm{T}\mathbb{R}^N|_M \to \mathcal{N}$ denote the projection of a vector onto the normal component to M fixing the base point.

To compare the (speed of the) flows in the tangential and normal directions, we need the following maps defined for every $p \in M$:

$$
\begin{aligned}
A_t(p) &= D\phi_{-t}(p)|_M \colon \mathrm{T}_pM \to \mathrm{T}_{\phi_{-t}(p)}M, \\
B_t(p) &= \Pi \circ D\phi_t(\phi_{-t}(p))|_M \colon \mathcal{N}_{\phi_{-t}(p)} \to \mathcal{N}_p.
\end{aligned}
$$

Here A_t is the linearization of the (backward) tangential flow, and B_t is the linearization of the flow in the normal direction.

Some helpful notation:

$$
w_0 \in \mathcal{N}_p \text{ and } v_0 \in \mathrm{T}_pM,
$$

with the corresponding vectors obtained under the linearized flow based at the point $\phi_{-t}(p)$ denoted by

$$
w_{-t} = (\Pi \circ D\phi_{-t}(p))w_0 \text{ and } v_{-t} = D\phi_{-t}(p)v_0,
$$

Definition 11. Let $p \in M$. The *generalized Lyapunov-type numbers* are defined by

$$
\nu(p) = \inf \left\{ a \colon \frac{1}{\|w_{-t}\| a^t} \to 0 \text{ as } t \to \infty, \forall w_0 \in \mathcal{N}_p \right\} \tag{10}
$$

and if $\nu(p) < 1$,

$$
\sigma(p) = \inf \left\{ b \colon \frac{\|v_{-t}\|}{\|w_{-t}\|^b} \to 0 \text{ as } t \to \infty, \forall w_0 \in \mathcal{N}_p, v_0 \in \mathrm{T}_pM \right\}. \tag{11}
$$

Remark 7. $\nu(p)$ is a quantitative measure of stability, while a small $\sigma(p)$ implies that the normal direction of the dynamics dominates the tangent direction.

Lemma 6.

$$
\begin{aligned}
\nu(p) &= \limsup_{t \to \infty} \|B_t(p)\|^{1/t} \\
\sigma(p) &= \limsup_{t \to \infty} \frac{\log \|A_t(p)\|}{-\log \|B_t(p)\|} \text{ if } \nu(p) < 1.
\end{aligned} \tag{12}
$$

Example 7. Consider the linear ODE given by

$$
\dot{z} = \begin{pmatrix} \lambda & 0 \\ 0 & -\mu \end{pmatrix} z \text{ for } z \in \mathbb{R}^2 \text{ and } \lambda, \mu > 0. \tag{13}
$$

$M = \{z = (z_1, z_2)^\top \in \mathbb{R}^2 \colon z_1 \in (-1,1), z_2 = 0\}$ is an overflowing invariant manifold inducing the splitting

$$
\mathrm{T}\mathbb{R}^2|_M = \mathrm{T}M \oplus \mathcal{N}
$$

where the two-dimensional bundles are given by

$$
\begin{aligned}
\mathrm{T}M &= \{(z_1, 0) \times (\mathbb{R}, 0) \colon z_1 \in (-1,1)\}, \\
\mathcal{N} &= \{(z_1, 0) \times (0, \mathbb{R}) \colon z_1 \in (-1,1)\}.
\end{aligned} \tag{14}
$$

To calculate A_t and B_t, we need the (linearized) flow for Eq. 13 and the projection Π onto $\mathcal{N}$. Direct calculation yields

$$D\phi_t = \begin{pmatrix} e^{\lambda} & 0 \\ 0 & e^{-\mu} \end{pmatrix}, \quad \Pi = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{15}$$

This gives the tangential and normal dynamics

$$A_t = D\phi_{-t}|_M(p) = \begin{pmatrix} e^{-\lambda t} & 0 \\ 0 & 0 \end{pmatrix}$$
$$B_t = \Pi \circ D\phi_t|_M(\phi_{-t}(p)) = \begin{pmatrix} 0 & 0 \\ 0 & e^{-\mu t} \end{pmatrix}$$

Therefore, the generalized Lyapunov-type numbers are given by

$$\nu(p) = \limsup_{t\to\infty} \|e^{-\mu t}\|^{1/t} = e^{-\mu}$$
$$\sigma(p) = \lim \sup_{t\to\infty} \frac{\log \|A_t(p)\|}{-\log \|B_t(p)\|} = \frac{\log |e^{-\lambda t}|}{-\log |e^{-\mu t}|} = -\frac{\lambda}{\mu}.$$

An analogous condition will be needed if we are dealing with an inflowing invariant manifold, just that the time direction will be reversed and asymptotic stability holds in backward time. This requires a refined definition of generalized Lyapunov-type numbers.

As before, it is assumed that M is a compact, connected, invariant manifold of class C^r for some $r \geq 1$. Suppose there exists a continuous splitting

$$\mathrm{T}\mathbb{R}^N|_M = \mathcal{N}^u \oplus \mathrm{T}M \oplus \mathcal{N}^s$$

with the projections

$$\Pi^s \colon \mathrm{T}\mathbb{R}^N|_M \to \mathcal{N}^s$$
$$\Pi^u \colon \mathrm{T}\mathbb{R}^N|_M \to \mathcal{N}^u.$$

Assume that the subbundles $\mathrm{T}M \oplus \mathcal{N}^s$ and $\mathrm{T}M \oplus \mathcal{N}^u$ are invariant under $D\phi_t$. Let $u_0 \in \mathcal{N}_p^u, v_0 \in \mathrm{T}_pM$ and $w_0 \in \mathcal{N}_p^s$ Let

$$u_{-t} = \Pi^u \circ D\phi_{-t}u_0, \text{ and } v_{-t} = D\phi_{-t}(p))v_0 \text{ and } w_{-t} = (\Pi \circ D\phi_{-t}(p))w_0$$

the images of the linearized flow.

Definition 12. The generalized Lyapunov-type numbers are

$$\nu^u(p) = \inf \left\{ a\colon \frac{\|u_{-t}\|}{a^t} \to 0 \text{ as } t \to \infty, \forall u_0 \in \mathcal{N}_p^u \right\}$$
$$\nu^s(p) = \inf \left\{ a\colon \frac{1}{\|w_{-t}\| a^t} \to 0 \text{ as } t \to \infty, \forall w_0 \in \mathcal{N}_p^s \right\} \tag{16}$$

and if $\nu(p) < 1$

$$
\begin{aligned}
\sigma^u(p) &= \inf\left\{b \colon \|v_{-t}\|\|u_{-t}\|^b \to 0 \text{ as } t\to\infty, \forall u_0 \in \mathcal{N}^u_p, v_0 \in \mathrm{T}_pM\right\} \\
\sigma^s(p) &= \inf\left\{b \colon \frac{\|v_{-t}\|}{\|w_{-t}\|^b} \to 0 \text{ as } t\to\infty, \forall w_0 \in \mathcal{N}^s_p, v_0 \in \mathrm{T}_pM\right\}.
\end{aligned} \tag{17}
$$

We can rely on the following way to calculate the generalized Lyapunov-type numbers:

$$
\begin{aligned}
\nu^u(p) &= \limsup_{t\to\infty} \|\Pi^u \circ D\phi_{-t}(\phi_t(p))|_{\mathcal{N}^u}\|^{1/t}, \\
\nu^s(p) &= \limsup_{t\to\infty} \|\Pi^s \circ D\phi_{t}(\phi_{-t}(p))|_{\mathcal{N}^s}\|^{1/t}, \\
\sigma^u(p) &= \limsup_{t\to\infty} \frac{\log\|D\phi_t|_M(p)\|}{-\log\|\Pi^u \circ D\phi_{-t}(\phi_t(p))|_{\mathcal{N}^u}\|}, \\
\sigma^s(p) &= \limsup_{t\to\infty} \frac{\log\|D\phi_t|_M(p)\|}{-\log\|\Pi^s \circ D\phi_{t}(\phi_{-t}(p))|_{\mathcal{N}^s}\|}.
\end{aligned} \tag{18}
$$

Definition 13. Let M be a compact connected invariant manifold in $\mathbb{R}^N$. A splitting $\mathrm{T}\mathbb{R}^N|_M = \mathcal{N}^u \oplus \mathrm{T}M \oplus \mathcal{N}^s$ is called *hyperbolic* if $\nu^u(p) < 1, \nu^s(p) < 1$ for all $p \in M$.

Definition 14. Let M be a compact connected invariant manifold in $\mathbb{R}^N$. A splitting $\mathrm{T}\mathbb{R}^N|_M = \mathcal{N}^u \oplus \mathrm{T}M \oplus \mathcal{N}^s$ is called *normally hyperbolic* if $\nu^u(p) < 1, \nu^s(p) < 1, \sigma^u(p) < 1$, and $\sigma^s(p) < 1$ for all $p \in M$. If an invariant manifold M admits a normally hyperbolic splitting, then it is called a normally hyperbolic invariant manifold.

Lemma 8. GLTNs are constant on orbits.

This lemma implies that if we know $\nu(p)$, then we get a bound on the linearized flow in the normal direction. The same reasoning obviously applies to $\sigma(p)$ and a suitable combination of A_t and B_t.

Remark 8. The notion of a normally hyperbolic manifold is one of the most important concepts in the geometric theory of dynamical systems. It is helpful to keep the colloquial version of Definition 14 in mind, which says that a manifold is normally hyperbolic if the linearized flow in the normal direction dominates the linearized flow in the tangential direction.

5.5 Perturbing invariant manifolds

With estimates on the linearized tangential and normal flows in hand, the goal is to prove a perturbation result for invariant manifolds.

Definition 15. Let H and H^{pert} be two C^1 vector fields on $\mathbb{R}^N$, and let $\mathcal{K}$ be a compact set. Then we say that H is C^1 θ-close to H^{pert} (on $\mathcal{K}$) if

$$
\begin{aligned}
\sup_{z\in\mathcal{K}} \|H(z) - H^{\mathrm{pert}}(z)\| &\le \theta, \\
\sup_{z\in\mathcal{K}} \|DH(z) - DH^{\mathrm{pert}}(z)\| &\le \theta.
\end{aligned}
$$

The definition is just a different way of stating that the unperturbed and perturbed vector fields are C^1-close in the sup-norm.

Now we can state Fenichel's major perturbation result for overflowing invariant manifolds.

Theorem 9. *Consider*

$$\dot{z} = H(z) \text{ with } H \in C^r \text{ and } z \in \mathbb{R}^N. \tag{19}$$

Let M be a C^r compact connected manifold that is overflowing invariant under the flow ϕ_t defined by Eq. 19. Assume that

$$\nu(p) < 1 \text{ and } \sigma(p) < \frac{1}{r} \text{ for all } p \in M. \tag{20}$$

Then for every C^r vector field H^{pert} that is C^1 θ-close to H, with θ sufficiently small, there is a manifold M^{pert} that is overflowing invariant under H^{pert} and C^r-diffeomorphic to M.

For the issue of how large the perturbation may be with the overflowing invariant manifold still persisting, see [9].

6 Van der Pol

Definition 16. A periodic solution γ_ϵ of a fast–slow system is called a relaxation oscillation if it converges (with respect to Hausdorff distance) in the singular limit $\epsilon \to 0$ to a candidate γ_0 consisting of alternating fast and slow segments forming a closed loop.

Definition 17. A periodic solution γ_ϵ of a fast–slow system is called a simple relaxation oscillation if it converges in the singular limit $\epsilon \to 0$ to a candidate γ_0 consisting of alternating fast and slow segments in which jumps occur only at generic fold points and the drop points are normally hyperbolic.

6.1 Singularities

So far, we have focused on the case in which the critical set is a manifold consisting of regular points or the even stronger assumption that C_0 is normally hyperbolic. A large part of multiple time scale dynamics deals with loss of regularity and normal hyperbolicity.

Definition 18 (Fold point)**.** Suppose $p \in C_0$, so that $f(p, 0) = 0$ holds. Then p is a fold point if

$$(D_x f)(p, 0) \text{ is of rank } m - 1.$$

A fold point is called nondegenerate if for vectors w and v, which are in the left and right nullspaces of $(D_x f)(p, 0)$ respectively, one has

$$w \cdot [(D_{xx} f)(p, 0)(v, v)] \neq 0 \text{ and } w \cdot [(D_y f)(p, 0)] \neq 0.$$

Example 10 (Fold bifurcation). The simplest example in which $(D_x f)(p, 0)$ is rank deficient is a (1, 1)-fast–slow system

$$\begin{aligned} x' &= y - x^2, \\ y' &= \epsilon g(x, y, \epsilon), \end{aligned} \tag{21}$$

with the critical manifold being a parabola

$$C_0 = \{(x, y) \in \mathbb{R}^2 \colon y = x^2\}.$$

We consider the origin $(x, y) = (0, 0)$ for which

$$(D_x f)(0, 0, 0) = \frac{\partial f}{\partial x}(0, 0, 0) = -2x|_{x=0} = 0. \tag{22}$$

Therefore, $(0, 0) \in C_0$ is not regular and not normally hyperbolic. Observe that the nondegeracy condition is satisfied':

$$\frac{\partial^2 f}{\partial\ x^2}(0, 0, 0) = f_{xx}(0, 0, 0) \neq 0. \tag{23}$$

Remark: two nonnormally hyperbolic points in Example 3.2.3 satisfy conditions analogous to Eqs. 22 and 23.

The fast subsystem of Eq. 21 is

$$\begin{aligned} x' &= y - x^2, \\ y' &= 0. \end{aligned} \tag{24}$$

In this case, $y \in \mathbb{R}$ is a parameter and Eq. 24 is the normal form for a fold bifurcation at $y = 0$; alternative terms for a fold bifurcation are saddle-node bifurcation, turning point, and limit point.

7 Definitions

7.1 General

Definition 19 (Compact)**.** A topological space X is called compact if every open cover of X has a finite subcover. For $\mathbb{R}^n$ compactness means closed and bounded.

Definition 20 (Diffeomorphism)**.** A diffeomorphism is a smooth and invertible function with smooth inverse.

Definition 21. Hausdorff distance between two nonempty sets $V, W \subset \mathbb{R}^n$, which is defined by

$$d_H(V, W) \coloneqq \max\left\{\sup_{v\in V}\inf_{w\in W}\|v - w\|, \sup_{w\in W}\inf_{v\in V}\|v - w\|\right\}.$$

7.2 Derivative

For a function $F\colon \mathbb{R}^m \to \mathbb{R}^n$, the total derivative $DF\colon \mathbb{R}^m \to L(\mathbb{R}^m, \mathbb{R}^n)$ assigns to each point $p \in \mathbb{R}^m$ a linear map $D_pF \in L(\mathbb{R}^m, \mathbb{R}^n)$. With respect to to the standard bases, this linear map can be represented as a matrix:

$$D_pF = \begin{pmatrix} \frac{\partial F_1}{\partial x_1}|_p & \cdots & \frac{\partial F_1}{\partial x_m}|_p \\ \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1}|_p & \cdots & \frac{\partial F_n}{\partial x_m}|_p \end{pmatrix}$$

This encodes the idea of "rate of change".

7.3 The tangent vector and bundle

A tangent vector at a point x on a smooth manifold M is a linear map $v\colon C^\infty(M) \to \mathbb{R}$ that satisfies the Leibniz rule, where $C^\infty(M)$ is the set of smooth functions on the manifold. In other words, a tangent vector is an operator that assigns a real number to each smooth function on the manifold, and this assignment is linear and obeys the product rule.

The *tangent space* $T_p(M)$ at a point p on a manifold M is the vector space consisting of all tangent vectors at that point. A tangent vector at p can be thought of as an equivalence class of smooth curves passing through p, where two curves are considered equivalent if their derivatives at p are equal.

$$\mathrm{T}M = \bigcup_{p \in M} \{p\} \times \mathrm{T}_pM$$

collects all the tangent spaces, i.e., elements in $\mathrm{T}M$ are pairs (p, T_pM) consisting of a point $p \in M$ and its associated tangent space. Since $M \subset \mathbb{R}^N$, the Euclidean inner product of the ambient space induces a splitting at each point $\mathrm{T}_p\,\mathbb{R}^N|_M = \mathrm{T}_pM \oplus \mathcal{N}_p$, where $\mathcal{N}_p$ denotes the normal space to T_pM consisting of all vectors orthogonal to T_pM and $\oplus$ is the usual direct sum. Hence, there is also a splitting $\mathrm{T}\mathbb{R}^N|_M = \mathrm{T}M \oplus \mathcal{N}$, where $\mathcal{N}$ denotes the normal bundle, which collects all normal spaces. Using the Euclidean inner product, we also have an associated norm $\|\cdot\|$ measuring the length of vectors in the tangent and normal bundles.

7.3.1 Tangent space as directional derivatives

One to one correspondance between vectors (thought of tangent vectors at a point) and derivations at a point.

Transversality

Definition 22 (Transversal intersection)**.** Two submanifolds M_1 and M_2 of a manifold M are transversal (or intersect transversally) in $\mathbb{R}^N$ if the tangent spaces T_pM_1 and T_pM_2 span $\mathrm{T}_p\,\mathbb{R}^N \simeq \mathbb{R}^N$ at each point $p \in M_1 \cap M_2$, i.e.,

$$\mathrm{T}_pM_1 \oplus \mathrm{T}_pM_2 = \mathrm{T}_pM.$$

We write $M_1 \pitchfork M_2$.

Remark 9. When two submanifolds or curves intersect in a transversal way within a manifold, it means that their respective tangent spaces intersect at a point without being tangent to each other. In other words, they cross at an angle, and they are not "parallel" or "tangent" in the traditional sense.

References

[1] N. Fenichel. Geometric singular perturbation theory for ordinary differential equations. *Journal of differential equations*, 31(1):53–98, 1979.

[2] C. K. R. T. Jones. Geometric singular perturbation theory. In L. Arnold, C. K. R. T. Jones, K. Mischaikow, G. Raugel, and R. Johnson, editors, *Dynamical Systems: Lectures Given at the 2nd Session of the Centro Internazionale Matematico Estivo (C.I.M.E.) held in Montecatini Terme, Italy, June 13–22, 1994*, pages 44–118. Springer Berlin Heidelberg, Berlin, Heidelberg, 1995. ISBN 9783540494157. doi: 10.1007/BFb0095239. URL https://doi.org/10.1007/BFb0095239.

[3] C. Kuehn. *Multiple time scale dynamics*, volume 191. Springer, 2015.

[4] J. Palis and S. Smale. Structural stability theorems. In *Global Analysis*, pages 223–231. Proc. Symp. Pure Math, 1970.

[5] M. C. Peixoto and M. M. Peixoto. Structural stability in the plane with enlarged boundary conditions. *An. Acad. Brasil. Ci*, 31(2):135–160, 1959.

[6] M. I. Rabinovich, R. Huerta, P. Varona, and V. S. Afraimovich. Transient cognitive dynamics, metastability, and decision making. *PLoS computational biology*, 4(5):e1000072, 2008.

[7] J. W. Robbin. A structural stability theorem. *Annals of Mathematics*, 94 (3):447–493, 1971.

[8] C. R. Robinson. Structural stability of C^1 flows. In *Dynamical Systems—Warwick 1974: Proceedings of a Symposium Held at the University of Warwick 1973/74*, pages 262–275. Springer, 1974.

[9] S. Wiggins. *Normally hyperbolic invariant manifolds in dynamical systems*, volume 105. Springer Science & Business Media, 1994.